International Journal of Scientific & Engineering Research Volume 9, Issue 8, August-2018
ISSN 2229-5518

Weak and Strong Convergence Results for Asymptotically Hemicontractive Mappings in Hilbert Spaces

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Abstract
We extend the results of Rafiq (A. Rafiq, On Mann Iteration in Hilbert Spaces, *Nonlinear Analysis* **66** (2007) 2230-2236) concerning the convergence theorems of hemicontractive mappings in Hilbert spaces to the more general class of asymptotically hemicontractive mappings.We modify the iteration scheme due to Rafiq (A. Rafiq, On Mann Iteration in Hilbert Spaces, *Nonlinear Analysis* **66** (2007) 2230-2236) and prove weak and strong convergence theorems for the class of asymptotically hemicontraction mappings, without the compactness condition imposed on the domain of the mappings. Our results compliment and extend the results of Rafiq (A. Rafiq, On Mann Iteration in Hilbert Spaces, *Nonlinear Analysis* **66** (2007) 2230-2236).

Keywords: Hemicontractive Mappings, Asymptotically Hemicontractive Mappings, Fixed Points, Hilbert Spaces

Introduction
Let H be a real Hilbert space. A mapping $T : H \to H$ is called:
(i) Nonexpansive(see eg [13]) if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in H$
(ii) Uniformly L-Lipschitzian(see eg [14]) if there exist a real constant $L > 0$, such that $||T^n x - T^n y|| \leq L||x - y||$, for all $x, y \in H,\ n \geq 1$
(iii) Pseudocontractive (see eg[5]) if $||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2$, for all $x, y \in H$
(iv) Asymptotically pseudocontractive (see eg[16]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $limk_n = 1$ and $||T^n x - T^n y||^2 \leq k_n||x - y||^2 + ||(I - T^n)x - (I - T^n)y||^2$, for all $x, y \in H,\ n \geq 1$
(v) Hemicontractive (see eg [1]) if $F(T) := \{x \in H : Tx = x\} \neq \emptyset$ and

$$||Tx - Ty||^2 \leq ||x - p||^2 + ||x - Tx||^2, \quad (1)$$

for all $x \in H,\ p \in F(T)$
(vi) Asymptotically hemicontractive (see eg [16]) if $F(T) := \{x \in H : Tx = x\} \neq \emptyset$, there exists a sequence $\{k_n\} \subset [1, \infty)$ with $limk_n = 1$ and

$$||T^n x - T^n y||^2 \leq k_n||x - p||^2 + ||x - T^n x||^2, \quad (2)$$

for all $x \in H,\ p \in F(T),\ n \geq 1$

It is easily seen that if a pseudocontractive mapping has a nonempty fixed point set, then it is a hemicontraction. Hence the class of pseudocontracion mappings with a nonempty fixed-points set is a subclass of the class of hemicontractive mappings.
It is also easily seen that every hemicontractive mapping is an asymptotically hemicontractive mapping with $n = 1$ and the sequence $k_n = 1$. Hence the class of hemicontractions is a subclass of the class of asymptotically hemicontractive mappings.
Observe that if an asymptotically pseudocontractive mapping mapping possesses a nonempty fixed-points set, then it is an asymptotically hemicontrative mapping. Hence, the class of asymptotically pseudocontractive mappings which possesses a nonempty fixed-points set is a subclass of the class of asymptotically hemicontractive mappings. Examples of asymptotically hemicontractive mappings are given in [24].

Definition 1 (see eg [25]): Let X and Y be metric spaces. A mapping $T : D(T) \subseteq X \to R(T) \subseteq Y$ is said to be completely continuous if it is continuous and maps bounded sets in $D(T)$ to pre-compact sets in $R(T)$, where $D(T)$ and $R(T)$ denote domain of T and range of T respectively.

Definition 2 (see eg [15]): A mapping $T : D(T) \subseteq X \to R(T) \subseteq Y$ is said to be demiclosed at a point p if whenever $\{x_n\} \subset D(T) \subseteq X$ is a sequence such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to $p \in R(T)$, then $Tx = p$.

In the recent past, many authors (see eg [1]-[2], [4]-[16], [18]-[24]) have studied existence and convergence results of fixed points of nonexpansive mappings and their generalizations, amongst which are pseudocontractions, hemicontractions and asymptotically hemicontractions. In order to obtain the existence and convergence results, authors (see eg [1]) have placed compactness, compactness-type and several other conditions on the domain of the operator or on the operator itself.
The construction of fixed points of nonexpansive mappings and their generalizations is achieved through iterative search techniques amonst which are the Mann, Mann-type, Ishikwa and Ishikawa-type schemes. Let X be a linear space and $T : X \to X$ be a map. The Mann iteration scheme (see e.g [17]) is the sequence generated from an arbitray $x_0 \in X$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

where $\{\alpha_n\}$ is a real nonnegative sequence satisfying certain conditionss.

In 2007, Rafiq [1] studied the convergence to fixed points of hemicontractive mappings in Hilbert spaces using a Mann-type iteration scheme generated from an arbitrary $x_0 \in H$ by $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n$, where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, satisfying certain conditions. More precisely, the author stated and proved the following theorems:

Theorem 1 [1]: Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ be a hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying

$\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 1 [1]: Let $H,\ K\ T$ be as in Theorem 1 and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $P_K : H \to K$ be the projection operator of H onto K. Then the sequence $x_n = P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tx_n),\quad n \geq 1$ converges strongly to a fixed point of T.

We observe that the compactness assumption imposed on K is strong.

It is our purpose in this paper to extend the results of Rafiq [1], to the more general class of asymptotically hemicontractive mappings. Furthermore, we prove related results for this class of mappings without imposing the compactness assumption on the domain of T. Our results generalize and compliment the results of Rafiq [1].

Before we state and prove our main results, we give some lemmas which will be useful in the sequel:

Lemma 1 [23]: Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that for some real number $N_0 \geq 1$, we have $a_{n+1} \leq a_n + c_n$. If:

(a) $\sum c_n < \infty$, then $\lim a_n$ exists

(b) $\sum c_n < \infty$, and $\{a_n\}$ has a subsequence which converges to zero, then $\lim a_n = 0$.

Lemma 2 (see[3], [11]): Let H be a real Hilbert space. Then for all $x, y \in H$, and $\lambda \in [0,1]$ the following well-known identity holds:

$$||(1-\lambda)x + \lambda y||^2 = (1-\lambda)||x||^2 + \lambda||y||^2 - \lambda(1-\lambda)||x-y||^2 \tag{3}$$

Lemma 3 [24]: Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping. Then $I - T$ is demiclosed at 0.

Main Results

We now state and prove our main results.

Lemma 4: Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a uniformly L- Lipschitzian asymptotically hemicontractive mapping. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T^n x_n, \tag{4}$$

where $\{k_n\}$ and $\{\alpha_n\}$ are real sequences in $[1,\infty]$ and $[0,1]$ respectively, satisfying:

(i) $0 < a \leq \alpha_n \leq b < 1$ for some real constants $a, b \in (0,1)$

(ii) $\sum(k_n - 1) < \infty$

Then

(a) $\lim ||x_n - p||$ *exists, where* $p \in F(T) := \{x \in C : Tx = x\}$

(b) $\lim ||x_n - Tx_n|| = 0$

Proof: Observe that $1-(1-\alpha_n)k_n > 0$. Otherwise, $1-(1-\alpha_n)k_n \leq 0$, which yields $(1-a)k_n \geq (1-\alpha_n)k_n \geq 1$. Taking limits as $n \to \infty$, we have $1-a \geq 1$, which is an absurdity since $a \in (0,1)$.

Now, let $p \in F(T)$. Then using (3), (4) and the fact that T is asymptotically hemicontractive, we have

$$\begin{aligned}
||x_n - p||^2 &= ||\alpha_n(x_{n-1}-p) + (1-\alpha_n)(T^n x_n - p)||^2 \\
&= \alpha_n||x_{n-1}-p||^2 + (1-\alpha_n)||T^n x_n - p||^2 - \alpha_n(1-\alpha_n)||x_{n-1} - T^n x_n||^2 \\
&\leq \alpha_n||x_{n-1}-p||^2 + (1-\alpha_n)[k_n||x_n - p||^2 \\
&\quad + ||x_n - T^n x_n||^2] - \alpha_n(1-\alpha_n)||x_{n-1} - T^n x_n||^2 \\
&= \alpha_n||x_{n-1}-p||^2 + (1-\alpha_n)k_n||x_n - p||^2 \\
&\quad + {\alpha_n}^2(1-\alpha_n)||x_{n-1} - T^n x_n||^2] - \alpha_n(1-\alpha_n)||x_{n-1} - T^n x_n||^2 \\
&= \alpha_n||x_{n-1}-p||^2 + (1-\alpha_n)k_n||x_n - p||^2 - \alpha_n(1-\alpha_n)^2||x_{n-1} - T^n x_n||^2 \\
&= \alpha_n||x_{n-1}-p||^2 + (1-\alpha_n)k_n||x_n - p||^2 - (1-\alpha_n)^2||x_n - T^n x_n||^2
\end{aligned}$$

This implies

$$[1-(1-\alpha_n)k_n]||x_n - p||^2 \leq \alpha_n||x_{n-1}-p||^2 - (1-\alpha_n)^2||x_n - T^n x_n||^2$$

Hence

$$\begin{aligned}
||x_n - p||^2 &\leq \frac{\alpha_n}{[1-(1-\alpha_n)k_n]}||x_{n-1}-p||^2 - \frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 \\
&= [1 + \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}]||x_{n-1}-p||^2 - \frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 \qquad (5)
\end{aligned}$$

Since C is a bounded subset of H, then $\{x_n\}$ and $\{||x_n - p||\}$ are all bounded. Hence there exists a real constant D such that $||x_n - p|| \leq D$. From (5), we have

$$||x_n - p||^2 \leq ||x_{n-1}-p||^2 + \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2 - \frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2. \qquad (6)$$

Letting $M = sup\{k_n\}$ and using lemma $4(i)$ and (6), we have

$$||x_n - p||^2 \leq ||x_{n-1}-p||^2 + \frac{(k_n-1)}{1-(1-a)M}D^2$$

Hence, from lemma 1, using lemma $4(ii)$, we have $\lim ||x_n - p||$ exists.

From (6) again, we have

$$\frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 - \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2 \leq ||x_{n-1}-p||^2 - ||x_n - p||^2 \qquad (7)$$

From (7), two cases arise:

(i) If

$$\frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 - \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2 \leq 0,$$

then we have

$$\frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 \leq \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2.$$

Hence

$$(1-b)||x_n - T^n x_n||^2 \leq (k_n - 1)D^2.$$

This yields

$$\lim ||x_n - T^n x_n||^2 = 0$$

(ii) If

$$\frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 - \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2 > 0,$$

then we have

$$\sum[\frac{(1-\alpha_n)^2}{[1-(1-\alpha_n)k_n]}||x_n - T^n x_n||^2 - \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}D^2] \leq ||x_0 - p||^2.$$

Letting $m = inf\{k_n\},\ M = sup\{k_n\}$, and using conditions (i)-(ii) of lemma 4, we have

$$\sum[\frac{(1-b)^2}{[1-(1-b)m]}||x_n - T^n x_n||^2 - \frac{(1-a)(k_n-1)}{1-(1-a)M}D^2] \leq ||x_0 - p||^2.$$

This yields

$$\lim \frac{(1-b)^2}{[1-(1-b)m]}||x_n - T^n x_n||^2 = \lim \frac{(1-a)(k_n-1)}{1-(1-a)M}D^2.$$

This implies

$$\lim ||x_n - T^n x_n||^2 = 0.$$

So in all cases $\lim ||x_n - T^n x_n|| = 0$.
Next, we have

$$\begin{aligned} ||x_n - x_{n-1}|| &= (1-\alpha_n)||T^n x_n - x_{n-1}|| \\ &= \frac{(1-\alpha_n)}{\alpha_n}||T^n x_n - x_n|| \\ &\leq \frac{1}{a}||T^n x_n - x_n|| \ \to 0 \ as \ n \to \infty \end{aligned}$$

Now,

$$\begin{aligned}
||x_{n-1} - Tx_n|| &\leq ||x_{n-1} - T^n x_n|| + L||T^{n-1}x_n - x_n|| \\
&\leq ||x_{n-1} - T^n x_n|| + L[||T^{n-1}x_n - T^{n-1}x_{n-1}|| + ||T^{n-1}x_{n-1} - x_n||] \\
&\leq ||x_{n-1} - T^n x_n|| + L^2||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_n|| \\
&\leq ||x_{n-1} - T^n x_n|| + L^2||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_{n-1}|| + L||x_n - x_{n-1}|| \\
&= ||x_{n-1} - T^n x_n|| + (L + L^2)||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_{n-1}|| \\
&= \frac{1}{\alpha_n}||x_n - T^n x_n|| + (L + L^2)||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_{n-1}|| \\
&\leq \frac{1}{a}||x_n - T^n x_n|| + (L + L^2)||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_{n-1}|| \to 0, \ as\ n \to \infty
\end{aligned}$$

We now have

$$||x_n - Tx_n|| \ \leq \ ||x_n - x_{n-1}|| + ||x_{n-1} - Tx_n|| \to 0, \ as\ n \to \infty$$

Theorem 2: Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a uniformly L- Lipschitzian asymptotically hemicontractive mapping which is such that $I - T$ is demiclosed at zero. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in C$ by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T^n x_n, \tag{8}$$

where $\{k_n\}$ and $\{\alpha_n\}$ are real sequences in $[1, \infty]$ and $[0, 1]$ respectively, satisfying:
(i) $0 < a \leq \alpha_n \leq b < 1$ for some real constants $a, b \in (0, 1)$
(ii) $\sum(k_n - 1) < \infty$
converges weakly to a fixed point of T

Proof:
Since $\{x_n\}$ is bounded, then it possesses a subsequence $\{x_{n_k}\}$ which is weakly convergent to $z \in C$. Also, since $\lim ||x_{n_k} - Tx_{n_k}|| = 0$, using definition 2 we have that $z \in F(T)$. Now, since H is an Opial space, using a standard argument (see e.g [13]), we obtain that $\{x_n\}$ converges weakly to $z \in F(T)$.

Theorem 3: Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a completely continuous uniformly L- Lipschitzian asymptotically hemicontractive mapping. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in C$ by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T^n x_n, \tag{9}$$

where $\{k_n\}$ and $\{\alpha_n\}$ are real sequences in $[1, \infty]$ and $[0, 1]$ respectively, satisfying:
(i) $0 < a \leq \alpha_n \leq b < 1$ for some real constants $a, b \in (0, 1)$

(ii) $\sum(k_n - 1) < \infty$
converges strongly to a fixed point of T

Proof:
Since $\{x_n\}$ is bounded and T is completely continuous, then $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$which converges strongly. Hence $\{x_{n_k}\}$ converges strongly. Suppose $\lim\{x_{n_k}\} = z$. Then $\lim\{Tx_{n_k}\} = Tz$. Hence $\lim ||x_{n_k} - Tx_{n_k}|| = ||z - Tz|| = 0$, so that $z \in F(T)$. Lemma 1 and lemma $4(a)$ now imply $\lim ||x_n - z|| = 0$. This completes the proof of the theorem.

Theorem 4: Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a uniformly L- Lipschitzian asymptotically pseudocontractive mapping. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in C$ by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T^n x_n, \tag{10}$$

where $\{k_n\}$ and $\{\alpha_n\}$ are real sequences in $[1, \infty]$ and $[0, 1]$ respectively, satisfying:
(i) $0 < a \leq \alpha_n \leq b < 1$ for some real constants $a, b \in (0, 1)$
(ii) $\sum(k_n - 1) < \infty$
converges weakly to a fixed point of T

Proof:
Since $\{x_n\}$ is bounded, then it possesses a subsequence $\{x_{n_k}\}$ which is weakly convergent to $z \in C$. Also, since$I - T$ is demiclosed at 0 (see lemma 3) and $\lim ||x_{n_k} - Tx_{n_k}|| = 0$, using definition 2 we have that $z \in F(T)$. Now, since H is an Opial space, using a standard argument (see e.g [13]), we obtain that $\{x_n\}$ converges weakly to $z \in F(T)$.

Competing Interests: The authors declare that there are not competing interests surrounding this research.

Acknowledgements: We want to state that this research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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